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Transcendence of solutions of q-Airy equation.

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Abstract. In this paper, we prove transcendence of solutions of the iterated Riccati equations associated with q-Airy equation when q is not a root of unity. The same result is obtained for a certain q-Bessel equation. Previously, we studied them under a stronger assumption that q is a transcendental number.

1. Introduction

In his paper [**3**], the author studied transcendence of functions which satisfy the iterated Riccati equations associated with q-Airy equation,

$$y(q^2t) + qty(qt) - y(t) = 0,$$

when q is a transcendental number. In this paper, we introduce a proof of transcendence which requires only that q is not a root of unity. The iterated Riccati equations are obtained in the following way. Setting $z(t) = y(qt)/y(t)$, we obtain the following first-order q-difference equation,

$$z(qt) = \frac{-qtz(t) + 1}{z(t)}.$$

We call this the (difference) Riccati equation associated with q-Airy equation. By iterations, we can express $z(q^it)$ in terms of $z(t)$ such as

$$z(q^2t) = \frac{(q^3t^2 + 1)z(t) - q^2t}{-qtz(t) + 1}.$$

This is a q^2-difference equation of Riccati form. The result of transcendence mentioned above implies unsolvability of q-Airy equation in the Franke's Liouvillian sense (cf. S. Nishioka [**3, 4**]).

A solution of the above Riccati equation satisfies q-Painlevé II equation of type $A_6^{(1)}$ (or $(A_1 + A_1')^{(1)}$), which is similar to the relations between Airy equation and Painlevé II equation. Moreover, each of the basic hypergeometric solutions of q-Airy equation has a limit to the Airy function (see Hamamoto, Kajiwara and Witte [**2**]).

The same result of transcendence is obtained for a q-Bessel equation

$$y(q^2t) + \left(\frac{t^2}{4} - q^\nu - q^{-\nu}\right) y(qt) + y(t) = 0$$

in the very same way introduced in this paper, where value of the parameter ν does not matter. This equation is related to one of the q-Bessel functions, $J_\nu^{(3)}(t;q)$. Here we set $y(qt) = J_\nu^{(3)}(tq^{\nu/2};q^2)$. For details of this function, see the book [**1**] by G. Gasper and M. Rahman.

Notation. Throughout the paper every field is of characteristic zero. When K is a field and τ is an isomorphism of K into itself, namely an injective endomorphism, the pair $\mathcal{K} = (K, \tau)$ is called a difference field. We call τ the (transforming) operator and K the underlying field. For a difference field $\mathcal{K}$, K often denotes its underlying field. For $a \in K$, the element $\tau^n a \in K$ $(n \in \mathbb{Z})$, if it exists, is called the n-th transform of a and is sometimes denoted by a_n. If $\tau K = K$, we say that $\mathcal{K}$ is inversive. For an algebraic closure $\overline{K}$ of K, the transforming operator τ is extended to an isomorphism $\overline{\tau}$ of $\overline{K}$ into itself, not necessarily in a unique way. We call the difference field $(\overline{K}, \overline{\tau})$ an algebraic closure of $\mathcal{K}$. For $p \in \mathbb{Z}_{>0}$, $\mathcal{K}^{(p)}$ denotes the difference field (K, τ^p). For difference fields $\mathcal{K} = (K, \tau)$ and $\mathcal{K}' = (K', \tau')$, $\mathcal{K}'/\mathcal{K}$ is called a difference field extension if K'/K is a field extension and $\tau'|_K = \tau$. In this case, we say that $\mathcal{K}'$ is a difference overfield of $\mathcal{K}$ and that $\mathcal{K}$ is a difference subfield of $\mathcal{K}'$. For brevity we sometimes use (K, τ') instead of $(K, \tau'|_K)$. We define a difference intermediate field in the proper way. Let $\mathcal{K}$ be a difference field, $\mathcal{L} = (L, \tau)$ a difference overfield of $\mathcal{K}$ and B a subset of L. The difference subfield $\mathcal{K}\langle B\rangle_{\mathcal{L}}$ of $\mathcal{L}$ is defined to be the difference field $(K(B, \tau B, \tau^2 B, \dots), \tau)$ and is denoted by $\mathcal{K}\langle B\rangle$ for brevity. A solution of a difference equation over $\mathcal{K}$ is defined to be an element of some difference overfield of $\mathcal{K}$ which satisfies the equation.

We use the following lemma.

Lemma 1.1 (Lemma 8 in S. Nishioka [**3**]). *Let C be an algebraically closed field, $q \in C^\times$ not a root of unity, t a transcendental element over C, $F/C(t)$ a finite extension of degree n, and τ an isomorphism of F into itself over C sending t to qt. Then $F = C(x)$, $x^n = t$.*

2. Notation for difference Riccati equation

Let $\mathcal{K} = (K, \tau)$ be a difference field, and let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(K),$$

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau^{i-1}A)(\tau^{i-2}A)\cdots(\tau A)A \quad (i = 1, 2, \dots).$$

In this paper, $\mathrm{Eq}(A,i)/\mathcal{K}$ denotes the equation over $\mathcal{K}$,

$$y_i(c^{(i)}y + d^{(i)}) = a^{(i)}y + b^{(i)}.$$

We easily see the following.

Lemma 2.1. *If f is a solution of $Eq(A,k)/\mathcal{K}$ in a difference field extension $\mathcal{L}/\mathcal{K}$, $f \in \mathcal{L}$ is also a solution of $Eq(A,ki)/\mathcal{K}$ $(i = 1, 2, \dots)$.*

Lemma 2.2. *Let $B = A_k$ and $B_i = (\tau^{k(i-1)}B)(\tau^{k(i-2)}B)\cdots B$ $(i = 1, 2, \dots)$. Then $B_i = A_{ki}$.*

Lemma 2.3. *For any $k, l, m \in \mathbb{Z}_{>0}$,*

$$\begin{aligned} &f \in \mathcal{L} \text{ is a solution of } Eq(A_k, lm)/\mathcal{K}^{(k)} \\ \Longleftrightarrow\, &f \in \mathcal{L}^{(l)} \text{ is a solution of } Eq(A_{kl}, m)/\mathcal{K}^{(kl)}, \end{aligned}$$

where $\mathcal{L}$ is a difference overfield of $\mathcal{K}^{(k)}$.

3. Proof of transcendence

Let C be an algebraically closed field and t a transcendental element over C. Let $q \in C^\times$ and $\mathcal{K} = (C(t), \tau_q : t \mapsto qt)$.

It is easy to prove that the Riccati equation associated with q-Airy equation has no rational function solution, and that is one of the keys to transcendence.

Lemma 3.1. *The equation over $\mathcal{K}$, $y_1 y = -qty + 1$, has no solution in $C(t)$.*

Proof. We prove this by contradiction. Assume that there exists a solution $f \in C(t)$. Let $f = P/Q$, where $P, Q \in C[t] \setminus \{0\}$ are relatively prime. Then we obtain

$$\frac{P_1}{Q_1} \cdot \frac{P}{Q} = -qt\frac{P}{Q} + 1,$$

and so

$$P_1 P = -qtPQ_1 + Q_1 Q.$$

This implies $P \mid Q_1$ and $Q_1 \mid P$. Hence, we find $\deg P = \deg Q$. However, the above equation yields $2\deg P = 2\deg P + 1$, a contradiction. $\square$

Let

$$A = \begin{pmatrix} -qt & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(C(t))$$

and

$$A_i = \begin{pmatrix} a^{(i)} & b^{(i)} \\ c^{(i)} & d^{(i)} \end{pmatrix} = (\tau_q^{i-1}A)(\tau_q^{i-2}A)\cdots(\tau_q A)A \quad (i = 1, 2, \dots).$$

Then

$$A_2 = (\tau_q A)A = \begin{pmatrix} q^3t^2+1 & -q^2t \\ -qt & 1 \end{pmatrix},$$

and for $i \geq 2$,

$$A_i = (\tau_q A_{i-1})A = \begin{pmatrix} a_1^{(i-1)} & b_1^{(i-1)} \\ c_1^{(i-1)} & d_1^{(i-1)} \end{pmatrix} \begin{pmatrix} -qt & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -qta_1^{(i-1)} + b_1^{(i-1)} & a_1^{(i-1)} \\ -qtc_1^{(i-1)} + d_1^{(i-1)} & c_1^{(i-1)} \end{pmatrix}$$

and

$$\begin{aligned} A_i = (\tau_q^{i-1}A)A_{i-1} &= \begin{pmatrix} -q^it & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a^{(i-1)} & b^{(i-1)} \\ c^{(i-1)} & d^{(i-1)} \end{pmatrix} \\ &= \begin{pmatrix} -q^ita^{(i-1)} + c^{(i-1)} & -q^itb^{(i-1)} + d^{(i-1)} \\ a^{(i-1)} & b^{(i-1)} \end{pmatrix}. \end{aligned}$$

Hence we find

$$b^{(i)} = a_1^{(i-1)}, \quad c^{(i)} = a^{(i-1)}, \quad d^{(i)} = b^{(i-1)} = c_1^{(i-1)},$$

and so for $i \geq 3$,

$$a^{(i)} = -q^ita^{(i-1)} + c^{(i-1)} = -q^ita^{(i-1)} + a^{(i-2)}.$$

By induction, we easily see

$$a^{(i)} = (-1)^i q^{i(i+1)/2} t^i + (\text{ terms of deg} \leq i-2 \text{ }) \tag{1}$$

for all $i \geq 1$. This implies

$$c^{(i)} = (-1)^{i-1} q^{(i-1)i/2} t^{i-1} + (\text{ terms of deg} \leq i-3 \text{ }) \quad (i \geq 1), \tag{2}$$

$$b^{(i)} = (-1)^{i-1} q^{(i-1)(i+2)/2} t^{i-1} + (\text{ terms of deg} \leq i-3 \text{ }) \quad (i \geq 1) \tag{3}$$

and

$$d^{(i)} = \begin{cases} (-1)^{i-2} q^{(i-2)(i+1)/2} t^{i-2} + (\text{ terms of deg} \leq i-4 \text{ }) & (i \geq 2), \\ 0 & (i = 1). \end{cases} \tag{4}$$

LEMMA 3.2. *$Eq(A_k, 1)/\mathcal{K}^{(k)}$ has a unique solution $f^{(k)}$ of the form*

$$\sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i, \quad e_i \in C,\ e_1 \neq 0,$$

in $(\mathbb{C}((1/t)), \tau_k\colon 1/t \mapsto q^{-k}(1/t))$. Moreover, $f^{(1)} = f^{(2)} = f^{(3)} = \cdots$ holds.

PROOF. (Uniqueness) Suppose there exists a solution f of $\mathrm{Eq}(A_k, 1)/\mathcal{K}^{(k)}$ in $(\mathbb{C}((1/t)), \tau_k)$ which is expressed as

$$f = \sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i, \quad e_i \in C,\ e_1 \neq 0.$$

Then f satisfies

$$\tau_k(f)(c^{(k)} f + d^{(k)}) = a^{(k)} f + b^{(k)}.$$

The left side is

$$\tau_k(f)(c^{(k)} f + d^{(k)}) = \left(\sum_{i=1}^{\infty} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right) \tag{5}$$

and the right side is

$$a^{(k)} f + b^{(k)} = a^{(k)} \sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i + b^{(k)}. \tag{6}$$

Comparing the coefficients of $(1/t)^{-k+1}$, we obtain

$$0 = (-1)^k q^{k(k+1)/2} e_1 + (-1)^{k-1} q^{(k-1)(k+2)/2},$$

and so $e_1 = q^{-1}$. For $j \geq 2$, the coefficient of $(1/t)^{-k+j}$ of the formula (6) is

$$(-1)^k q^{k(k+1)/2} e_j + P_j,$$

where P_j is determined by $e_1, \ldots, e_{j-1}$. On the other hand, for $j \geq 2$, the coefficient of $(1/t)^{-k+j}$ of the formula (5) is equal to the coefficient of $(1/t)^{-k+j}$ of

$$\left(\sum_{i=1}^{j-1} \frac{e_i}{q^{ki}} \left(\frac{1}{t}\right)^i\right) \left(c^{(k)} \sum_{i=1}^{j-1} e_i \left(\frac{1}{t}\right)^i + d^{(k)}\right),$$

which is denoted by Q_j and also determined by $e_1, \ldots, e_{j-1}$. Hence we find

$$(-1)^k q^{k(k+1)/2} e_j = Q_j - P_j,$$

and so

$$e_j = (-1)^k q^{-k(k+1)/2}(Q_j - P_j),$$

which implies e_j is determined by $e_1, \dots, e_{j-1}$. Therefore we conclude that f is unique.

(Existence) Define e_i $(i = 1, 2, \dots)$ as

$$e_1 = q^{-1}, \quad e_i = (-1)^k q^{-k(k+1)/2}(Q_i - P_i) \quad (i \geq 2).$$

By the above discussion, it follows that

$$f^{(k)} = \sum_{i=1}^{\infty} e_i \left(\frac{1}{t}\right)^i$$

is a solution of $\mathrm{Eq}(A_k, 1)/\mathcal{K}^{(k)}$.

(Identity) Fix $k \geq 1$. Since $f^{(1)}$ is a solution of $\mathrm{Eq}(A, 1)/\mathcal{K}$ in $(\mathbb{C}((1/t)), \tau_1)$, it is a solution of $\mathrm{Eq}(A, k)/\mathcal{K}$ in $(\mathbb{C}((1/t)), \tau_1)$. Hence $f^{(1)}$ is a solution of $\mathrm{Eq}(A_k, 1)/\mathcal{K}^{(k)}$ in $(\mathbb{C}((1/t)), \tau_1^k = \tau_k)$. By the uniqueness, we find $f^{(k)} = f^{(1)}$. $\square$

Theorem 3.3. *Suppose q is not a root of unity. Then for any k, $Eq(A_k, 1)/\mathcal{K}^{(k)}$ has no solution algebraic over $C(t)$.*

Proof. We prove this by contradiction. Assume there exists k such that $\mathrm{Eq}(A_k, 1)/\mathcal{K}^{(k)}$ has a solution f algebraic over $C(t)$. Let $\mathcal{L} = (L, \tau) = \mathcal{K}^{(k)}\langle f \rangle$. Then f satisfies

$$\tau(f)(c^{(k)}f + d^{(k)}) = a^{(k)}f + b^{(k)}. \tag{7}$$

We obtain $\det A_k \neq 0$ from $\det A = -1$. Hence $c^{(k)}f + d^{(k)} \neq 0$, and so

$$\tau(f) = \frac{a^{(k)}f + b^{(k)}}{c^{(k)}f + d^{(k)}} \in C(t, f).$$

This means $L = C(t, f)$. Let $n = [L : C(t)]$ be the degree of the extension. By Lemma 1.1, we find $L = C(x)$, $x^n = t$. It follows that x is transcendental over C. By the calculation,

$$\left(\frac{\tau x}{x}\right)^n = \frac{\tau(x^n)}{x^n} = \frac{\tau t}{t} = \frac{\tau_q^k t}{t} = q^k,$$

we obtain $\tau x/x \in C^\times$. Let $r \in C^\times$ denote it. Then $\tau x = rx$ holds. Note $f \in C(x)^\times$ and $A_k \in \mathrm{M}_2(C[x^n])$. Expressing f as $f = P/Q$, where $P, Q \in C[x]$ are relatively prime, we obtain the following equation from the equation (7),

$$\frac{\tau P}{\tau Q} = \frac{a^{(k)}\frac{P}{Q} + b^{(k)}}{c^{(k)}\frac{P}{Q} + d^{(k)}} = \frac{a^{(k)}P + b^{(k)}Q}{c^{(k)}P + d^{(k)}Q}.$$

Since $\tau P, \tau Q$ are relatively prime, there exists $R \in C[x]$ such that

$$\begin{cases} R\tau(P) = a^{(k)}P + b^{(k)}Q, \\ R\tau(Q) = c^{(k)}P + d^{(k)}Q. \end{cases} \tag{8}$$

Noting $\det A_k = (-1)^k$, we can calculate as follows,

$$R\begin{pmatrix}\tau P \\ \tau Q\end{pmatrix} = \begin{pmatrix}a^{(k)} & b^{(k)} \\ c^{(k)} & d^{(k)}\end{pmatrix}\begin{pmatrix}P \\ Q\end{pmatrix},$$

$$(-1)^k R\begin{pmatrix}d^{(k)} & -b^{(k)} \\ -c^{(k)} & a^{(k)}\end{pmatrix}\begin{pmatrix}\tau P \\ \tau Q\end{pmatrix} = \begin{pmatrix}P \\ Q\end{pmatrix}.$$

Since P, Q are relatively prime, we find $R \in C^{\times}$. Comparing the degrees of the equation (8), we obtain

$$\deg_x(a^{(k)}P + b^{(k)}Q) = \deg_x(R\tau(P)) = \deg_x P.$$

Since $\deg_x a^{(k)} = kn \geq 1$,

$$\deg_x a^{(k)}P = \deg_x b^{(k)}Q,$$

which means

$$\deg_x Q - \deg_x P = \deg_x a^{(k)} - \deg_x b^{(k)} = kn - (k-1)n = n.$$

By this result, express f as

$$f = \sum_{i=n}^{\infty} e_i \left(\frac{1}{x}\right)^i, \quad e_i \in C,\ e_n \neq 0,$$

and extend the isomorphism $\tau\colon C(1/x) \to C(1/x)$ sending $1/x$ to $r^{-1}(1/x)$ to the isomorphism $\tau\colon C((1/x)) \to C((1/x))$ sending $1/x$ to $r^{-1}(1/x)$. We will show $f \in C(t)$. We prove that $n \nmid i$ implies $e_i = 0$ $(i \geq n)$ by contradiction. Assume there exists $i \geq n$ such that $n \nmid i$ and $e_i \neq 0$. Let $ln + m$ $(0 < m < n)$ be the minimum of such numbers. The first term of

$$\begin{aligned} &a^{(k)}f+b^{(k)} \\ &= a^{(k)}\left(e_n\left(\frac{1}{x}\right)^n+\cdots+e_{ln}\left(\frac{1}{x}\right)^{ln}+e_{ln+m}\left(\frac{1}{x}\right)^{ln+m}+\cdots\right)+b^{(k)} \end{aligned}$$

whose exponent is not divisible by n has the exponent

$$-kn+(ln+m).$$

On the other hand, the first term of

$$\begin{aligned} &\tau(f)(c^{(k)}f+d^{(k)}) \\ &= \left\{\frac{e_n}{r^n}\left(\frac{1}{x}\right)^n+\cdots+\frac{e_{ln}}{r^{ln}}\left(\frac{1}{x}\right)^{ln}+\frac{e_{ln+m}}{r^{ln+m}}\left(\frac{1}{x}\right)^{ln+m}+\cdots\right\} \\ &\times\left\{c^{(k)}\left(e_n\left(\frac{1}{x}\right)^n+\cdots+e_{ln}\left(\frac{1}{x}\right)^{ln}+e_{ln+m}\left(\frac{1}{x}\right)^{ln+m}+\cdots\right)+d^{(k)}\right\} \end{aligned}$$

whose exponent is not divisible by n has the exponent greater than or equal to

$$(2-k)n+(ln+m).$$

Hence we obtain

$$-kn+(ln+m)\geq(2-k)n+(ln+m),$$

a contradiction. We proved that $n\nmid i$ implies $e_i=0$ $(i\geq n)$, which means

$$f\in C(((1/x)^n))\cap C(1/x)=C((1/x)^n)=C(1/t)=C(t).$$

It follows from the above result that $L=C(t,f)=C(t)$ and $n=[L:C(t)]=1$. Hence we find $x=t$, $r=q^k$ and

$$f=\sum_{i=1}^{\infty}e_i\left(\frac{1}{t}\right)^i\in C(t),\quad e_i\in C,\ e_1\neq 0.$$

Since f is a solution of $\mathrm{Eq}(A_k,1)/\mathcal{K}^{(k)}$ in $(C((1/t)),\tau\colon 1/t\mapsto q^{-k}(1/t))$, f is a solution of $\mathrm{Eq}(A_1,1)/\mathcal{K}$ by Lemma 3.2. However, Lemma 3.1 says that $\mathrm{Eq}(A_1,1)/\mathcal{K}$ has no solution in $C(t)$. □

Remark 3.4. *Considering the proofs in the author's paper* [**3**] *or paper* [**4**], *we* *easily obtain the same theorem for q-Bessel equation,*

$$y(q^2t) + \left(\frac{t^2}{4} - q^{\nu} - q^{-\nu}\right) y(qt) + y(t) = 0,$$

in the very same way. This result is independent of value of the parameter ν*.*

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